

A NOTE ON BRIDGELAND'S HALL ALGEBRAS

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ABSTRACT. In this note, let \mathcal{A} be a finitary hereditary abelian category with enough projectives. By using the associativity formula of Hall algebras, we give a new and simple proof of the main theorem in [17], which states that the Bridgeland's Hall algebra of 2-cyclic complexes of projective objects in \mathcal{A} is isomorphic to the Drinfeld double Hall algebra of \mathcal{A} . In a similar way, we give a simplification of the key step in the proof of Theorem 4.11 in [5].

1. INTRODUCTION

Ringel [9] introduced the Hall algebra of a finite dimensional algebra over a finite field. By the works of Ringel [9, 10, 11] and Green [3], the twisted Hall algebra, called the Ringel–Hall algebra, of a finite dimensional hereditary algebra provides a realization of the positive (negative) part of the corresponding quantum group. In order to obtain a Hall algebra description of the entire quantum group, one considers the Hall algebras of triangulated categories (for example, [4], [14], [16]). In [15], Xiao gave a realization of the whole quantum group by constructing the Drinfeld double of the extended Ringel–Hall algebra of any hereditary algebra.

In 2013, Bridgeland [1] considered the Hall algebra of 2-cyclic complexes of projective modules over a finite dimensional hereditary algebra A , and achieved an algebra, called the (reduced) Bridgeland's Hall algebra of A , by taking some localization and reduction. He proved that there is an algebra embedding from the Ringel–Hall algebra of A to its Bridgeland's Hall algebra. Moreover, the quantum group associated with A can be embedded into the reduced Bridgeland's Hall algebra of A . This provides a realization of the full quantum group by Hall algebras. In [1], Bridgeland stated without proof that the Bridgeland's Hall algebra of each finite dimensional hereditary algebra is isomorphic to the Drinfeld double of its extended Ringel–Hall algebra. Later on, Yanagida proved this statement in [17]. With the purpose of generalizing Bridgeland's construction to a bigger class of exact categories, Gorsky [2] defined the so-called semi-derived Hall algebra of the category of bounded complexes of \mathcal{E} for each exact category \mathcal{E} satisfying certain finiteness conditions. In particular, if every object in \mathcal{E} has finite projective resolution, he gave a similar construction to the category of 2-cyclic complexes of \mathcal{E} . Recently, inspired

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by the works of Bridgeland and Gorsky, Lu and Peng [5] have generalized Bridgeland's construction to any hereditary abelian category \mathcal{A} which may not have enough projectives, and defined an algebra for the category of 2-cyclic complexes of \mathcal{A} , called the modified Ringel–Hall algebra of \mathcal{A} . They also proved that the resulting algebra is isomorphic to the Drinfeld double Hall algebra of \mathcal{A} .

The key step in the proof that the Bridgeland's Hall algebra or modified Ringel–Hall algebra of a hereditary abelian category \mathcal{A} is isomorphic to its Drinfeld double Hall algebra is to check the (Drinfeld) commutator relations. The method used in [17] is to make the summations on the left-hand and right-hand sides of the commutator relations symmetric by some analysis of the structure of the category of 2-cyclic complexes of projectives in \mathcal{A} , as well as some complicated calculations. It seems that the process of the proof is not too intuitive. While Lu and Peng gave a characterization of some coefficients in the commutator relations by introducing two sets, and obtained the proof by means of the coincidence of the cardinalities of these two sets. Nevertheless, their characterization is a bit complicated. In this note we use the associativity formula of Hall algebras to give a more intuitive and simpler proof that the Bridgeland's Hall algebra of \mathcal{A} is isomorphic to its Drinfeld double Hall algebra. Similarly, based on [5], we give a simplification of the key step in the proof of Theorem 4.11 in [5]. Explicitly, we prove the commutator relations therein by using the associativity formula of Hall algebras rather than Lemma 4.10 in [5].

Let us fix some notations used throughout the paper. k is always a finite field with q elements and set $v = \sqrt{q}$. \mathcal{A} is always a finitary hereditary abelian k -category with enough projectives unless otherwise stated, we also assume that the image \hat{M} of M in the Grothendieck group $K_0(\mathcal{A})$ is nonzero for any nonzero object M in \mathcal{A} (cf. [1, 17]). We denote by $\text{Iso}(\mathcal{A})$ the set of isoclasses (isomorphism classes) of objects in \mathcal{A} . The subcategory of \mathcal{A} consisting of projective objects is denoted by \mathcal{P} . For a complex $M_\bullet = (\cdots \rightarrow M_{i+1} \xrightarrow{d_{i+1}} M_i \rightarrow \cdots)$ in \mathcal{A} , its homology is denoted by $H_*(M_\bullet)$. For a finite set S , we denote by $|S|$ its cardinality. For an object $M \in \mathcal{A}$, we denote by $\text{Aut}_{\mathcal{A}}(M)$ the automorphism group of M , and set $a_M = |\text{Aut}_{\mathcal{A}}(M)|$.

2. PRELIMINARIES

In this section, we collect some necessary definitions and properties. All of the materials can be found in [1], [12] and [17].

2.1. 2-cyclic complexes. Let $\mathcal{C}_2(\mathcal{A})$ be the abelian category of 2-cyclic complexes over \mathcal{A} . The objects of this category consist of diagrams

$$M_\bullet = M_1 \begin{array}{c} \xrightarrow{d_1^M} \\ \xleftarrow{d_0^M} \end{array} M_0$$

in \mathcal{A} such that $d_{i+1}^M \circ d_i^M = 0$, $i \in \mathbb{Z}_2$. A morphism $s_\bullet : M_\bullet \rightarrow N_\bullet$ consists of a diagram

$$\begin{array}{ccc} M_1 & \xrightleftharpoons[d_0^M]{d_1^M} & M_0 \\ s_1 \downarrow & & \downarrow s_0 \\ N_1 & \xrightleftharpoons[d_0^N]{d_1^N} & N_0 \end{array}$$

with $s_{i+1} \circ d_i^M = d_i^N \circ s_i$, $i \in \mathbb{Z}_2$. Two morphisms $s_\bullet, t_\bullet : M_\bullet \rightarrow N_\bullet$ are said to be *homotopic* if there are morphisms $h_i : M_i \rightarrow N_{i+1}$, $i \in \mathbb{Z}_2$, such that $t_i - s_i = d_{i+1}^N \circ h_i + h_{i+1} \circ d_i^M$, $i \in \mathbb{Z}_2$. For an object $M_\bullet \in \mathcal{C}_2(\mathcal{A})$, we define its class in the Grothendieck group $K_0(\mathcal{A})$ to be

$$\hat{M}_\bullet := \hat{M}_0 - \hat{M}_1 \in K_0(\mathcal{A}).$$

Denote by $\mathcal{K}_2(\mathcal{A})$ the homotopy category obtained from $\mathcal{C}_2(\mathcal{A})$ by identifying homotopic morphisms. Denote by $\mathcal{C}_2(\mathcal{P}) \subset \mathcal{C}_2(\mathcal{A})$ the full subcategory whose objects are complexes of projectives in \mathcal{A} , and by $\mathcal{K}_2(\mathcal{P})$ its homotopy category. The shift functor of complexes induces an involution of $\mathcal{C}_2(\mathcal{A})$. This involution shifts the grading and changes the signs of differentials as follows

$$M_\bullet = M_1 \xrightleftharpoons[d_0^M]{d_1^M} M_0 \xrightarrow{*} M_\bullet^* = M_0 \xrightleftharpoons[-d_1^M]{-d_0^M} M_1.$$

Let $\mathcal{D}^b(\mathcal{A})$ be the bounded derived category of \mathcal{A} , with the suspension functor $[1]$. Let $\mathcal{R}_2(\mathcal{A}) = \mathcal{D}^b(\mathcal{A})/[2]$ be the orbit category, also known as the root category of \mathcal{A} . The category $\mathcal{D}^b(\mathcal{A})$ is equivalent to the bounded homotopy category $K^b(\mathcal{P})$, since \mathcal{A} is hereditary. In this case, we can equally well define $\mathcal{R}_2(\mathcal{A})$ as the orbit category of $K^b(\mathcal{P})$.

Lemma 2.1. ([7], [1, Lemma 3.1]) *There is an equivalence $D : \mathcal{R}_2(\mathcal{A}) \rightarrow \mathcal{K}_2(\mathcal{P})$ sending a bounded complex of projectives $(P_i)_{i \in \mathbb{Z}}$ to the 2-cyclic complex*

$$\bigoplus_{i \in \mathbb{Z}} P_{2i+1} \xrightleftharpoons{\quad} \bigoplus_{i \in \mathbb{Z}} P_{2i}.$$

Lemma 2.2. ([1, Lemma 3.3]) *If $M_\bullet, N_\bullet \in \mathcal{C}_2(\mathcal{P})$, then there exists an isomorphism of vector spaces*

$$\text{Ext}_{\mathcal{C}_2(\mathcal{A})}^1(N_\bullet, M_\bullet) \cong \text{Hom}_{\mathcal{K}_2(\mathcal{A})}(N_\bullet, M_\bullet^*).$$

A complex $M_\bullet \in \mathcal{C}_2(\mathcal{A})$ is called *acyclic* if $H_*(M_\bullet) = 0$. Each object $P \in \mathcal{P}$ determines acyclic complexes

$$K_P = (P \xrightleftharpoons[0]{1} P), \quad K_P^* = (P \xrightleftharpoons[1]{0} P).$$

Lemma 2.3. ([1, Lemma 3.2]) *For each acyclic complex $M_\bullet \in \mathcal{C}_2(\mathcal{P})$, there are objects $P, Q \in \mathcal{P}$, unique up to isomorphism, such that $M_\bullet \cong K_P \oplus K_Q^*$.*

2.2. Hall algebras. Given objects $L, M, N \in \mathcal{A}$, let $\text{Ext}_{\mathcal{A}}^1(M, N)_L \subset \text{Ext}_{\mathcal{A}}^1(M, N)$ be the subset consisting of those equivalence classes of short exact sequences with middle term L .

Definition 2.4. The *Hall algebra* $\mathcal{H}(\mathcal{A})$ of \mathcal{A} is the vector space over \mathbb{C} with basis elements $[M] \in \text{Iso}(\mathcal{A})$, and with multiplication defined by

$$[M] \diamond [N] = \sum_{[L] \in \text{Iso}(\mathcal{A})} \frac{|\text{Ext}_{\mathcal{A}}^1(M, N)_L|}{|\text{Hom}_{\mathcal{A}}(M, N)|} [L].$$

By [9], the above operation \diamond defines on $\mathcal{H}(\mathcal{A})$ the structure of a unital associative algebra over \mathbb{C} , and the class $[0]$ of the zero object is the unit.

Remark 2.5. Given objects $L, M, N \in \mathcal{A}$, set

$$g_{MN}^L = |\{N' \subset L \mid N' \cong N, L/N' \cong M\}|.$$

By Riedtmann-Peng formula [8, 6],

$$g_{MN}^L = \frac{|\text{Ext}_{\mathcal{A}}^1(M, N)_L|}{|\text{Hom}_{\mathcal{A}}(M, N)|} \cdot \frac{a_L}{a_M a_N}.$$

Thus in terms of alternative generators $[[M]] = \frac{[M]}{a_M}$, the product takes the form

$$[[M]] \diamond [[N]] = \sum_{[L] \in \text{Iso}(\mathcal{A})} g_{MN}^L [[L]],$$

which is the definition used, for example, in [9, 12]. The associativity of Hall algebras amounts to the following formula

$$\sum_{[M] \in \text{Iso}(\mathcal{A})} g_{XY}^M g_{MZ}^L = \sum_{[N] \in \text{Iso}(\mathcal{A})} g_{XN}^L g_{YZ}^N (= g_{XYZ}^L), \quad (2.1)$$

for any objects $L, X, Y, Z \in \mathcal{A}$.

For objects $M, N \in \mathcal{A}$, let

$$\langle M, N \rangle := \dim_k \text{Hom}_{\mathcal{A}}(M, N) - \dim_k \text{Ext}_{\mathcal{A}}^1(M, N),$$

and it descends to give a bilinear form

$$\langle \cdot, \cdot \rangle : K_0(\mathcal{A}) \times K_0(\mathcal{A}) \longrightarrow \mathbb{Z},$$

known as the *Euler form*. We also consider the *symmetric Euler form*

$$(\cdot, \cdot) : K_0(\mathcal{A}) \times K_0(\mathcal{A}) \longrightarrow \mathbb{Z},$$

defined by $(\alpha, \beta) = \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle$ for all $\alpha, \beta \in K_0(\mathcal{A})$. The *Ringel–Hall algebra* $\mathcal{H}_{\text{tw}}(\mathcal{A})$ of \mathcal{A} is the same vector space as $\mathcal{H}(\mathcal{A})$, but with multiplication defined by

$$[M] * [N] = v^{\langle \hat{M}, \hat{N} \rangle} \cdot [M] \diamond [N].$$

The *extended Ringel–Hall algebra* $\tilde{\mathcal{H}}(\mathcal{A})$ of \mathcal{A} is defined as an extension of $\mathcal{H}_{\text{tw}}(\mathcal{A})$ by adjoining symbols K_α for $\alpha \in K(\mathcal{A})$, and imposing relations

$$K_\alpha K_\beta = K_{\alpha+\beta}, \quad K_\alpha[M] = v^{\langle \alpha, \hat{M} \rangle} [M] K_\alpha,$$

for $\alpha, \beta \in K_0(\mathcal{A})$ and $[M] \in \text{Iso}(\mathcal{A})$.

By Green [3] and Xiao [15], the extended Ringel–Hall algebra $\tilde{\mathcal{H}}(\mathcal{A})$ is a topological bialgebra (see [12]) with comultiplication Δ and counit ϵ defined by

$$\Delta([L]K_\alpha) = \sum_{[M], [N] \in \text{Iso}(\mathcal{A})} v^{\langle \hat{M}, \hat{N} \rangle} g_{MN}^L [M] K_{\alpha+\hat{N}} \otimes [N] K_\alpha \quad \text{and} \quad \epsilon([L]K_\alpha) = \delta_{L,0}.$$

It is well known that there exists a nondegenerate symmetric bilinear

$$\varphi(-, -) : \tilde{\mathcal{H}}(\mathcal{A}) \times \tilde{\mathcal{H}}(\mathcal{A}) \longrightarrow \mathbb{C},$$

defined by

$$\varphi([M]K_\alpha, [N]K_\beta) = \delta_{[M], [N]} a_M v^{(\alpha, \beta)}.$$

This is a Hopf pairing (see for example [3, 12, 15]). Then the *Drinfeld double Hall algebra* $D(\mathcal{A})$ of \mathcal{A} is by definition the free product $\tilde{\mathcal{H}}(\mathcal{A}) * \tilde{\mathcal{H}}(\mathcal{A})$ divided out by the commutator relations (with $a, b \in \tilde{\mathcal{H}}(\mathcal{A})$)

$$\sum \varphi(a_{(2)}, b_{(1)}) \cdot a_{(1)} \otimes b_{(2)} = \sum \varphi(a_{(1)}, b_{(2)}) (1 \otimes b_{(1)}) \circ (a_{(2)} \otimes 1). \quad (2.2)$$

Here we use Sweedler's notation: $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$.

2.3. Bridgeland's Hall algebras. Let $\mathcal{H}(\mathcal{C}_2(\mathcal{A}))$ be the Hall algebra of the abelian category $\mathcal{C}_2(\mathcal{A})$ defined in Definition 2.4 and $\mathcal{H}(\mathcal{C}_2(\mathcal{P})) \subset \mathcal{H}(\mathcal{C}_2(\mathcal{A}))$ be the subspace spanned by the isoclasses of complexes of projective objects. Define $\mathcal{H}_{\text{tw}}(\mathcal{C}_2(\mathcal{P}))$ to be the same vector space as $\mathcal{H}(\mathcal{C}_2(\mathcal{P}))$ with “twisted” multiplication defined by

$$[M_\bullet] * [N_\bullet] := v^{\langle \hat{M}_0, \hat{N}_0 \rangle + \langle \hat{M}_1, \hat{N}_1 \rangle} \cdot [M_\bullet] \diamond [N_\bullet].$$

Then $\mathcal{H}_{\text{tw}}(\mathcal{C}_2(\mathcal{P}))$ is an associative algebra (see [1]).

We have the following simple relations for the acyclic complexes K_P and K_P^* .

Lemma 2.6. ([1, Lemma 3.5]) *For any object $P \in \mathcal{P}$ and any complex $M_\bullet \in \mathcal{C}_2(\mathcal{P})$, we have the following relations in $\mathcal{H}_{\text{tw}}(\mathcal{C}_2(\mathcal{P}))$*

$$[K_P] * [M_\bullet] = v^{\langle \hat{P}, \hat{M}_\bullet \rangle} [K_P \oplus M_\bullet], \quad [M_\bullet] * [K_P] = v^{-\langle \hat{M}_\bullet, \hat{P} \rangle} [K_P \oplus M_\bullet]; \quad (2.3)$$

$$[K_P^*] * [M_\bullet] = v^{-\langle \hat{P}, \hat{M}_\bullet \rangle} [K_P^* \oplus M_\bullet], \quad [M_\bullet] * [K_P^*] = v^{\langle \hat{M}_\bullet, \hat{P} \rangle} [K_P^* \oplus M_\bullet]; \quad (2.4)$$

$$[K_P] * [M_\bullet] = v^{\langle \hat{P}, \hat{M}_\bullet \rangle} [M_\bullet] * [K_P], \quad [K_P^*] * [M_\bullet] = v^{-\langle \hat{P}, \hat{M}_\bullet \rangle} [M_\bullet] * [K_P^*]. \quad (2.5)$$

In particular, for $P, Q \in \mathcal{P}$, we have

$$[K_P] * [K_Q] = [K_P \oplus K_Q], \quad [K_P] * [K_Q^*] = [K_P \oplus K_Q^*]; \quad (2.6)$$

$$[[K_P], [K_Q]] = [[K_P], [K_Q^*]] = [[K_P^*], [K_Q^*]] = 0. \quad (2.7)$$

By Lemmas 2.3 and 2.6, the acyclic elements of $\mathcal{H}_{\text{tw}}(\mathcal{C}_2(\mathcal{P}))$ satisfy the Ore conditions and thus we have the following definition from [1].

Definition 2.7. The *Bridgeland's Hall algebra* of \mathcal{A} , denoted by $\mathcal{DH}(\mathcal{A})$, is the localization of $\mathcal{H}_{\text{tw}}(\mathcal{C}_2(\mathcal{P}))$ with respect to the elements $[M_\bullet]$ corresponding to acyclic complexes M_\bullet . In symbols,

$$\mathcal{DH}(\mathcal{A}) := \mathcal{H}_{\text{tw}}(\mathcal{C}_2(\mathcal{P}))[[M_\bullet]^{-1} \mid H_*(M_\bullet) = 0].$$

As explained in [1], this is the same as localizing by the elements $[K_P]$ and $[K_P^*]$ for all objects $P \in \mathcal{P}$. Writing $\alpha \in K_0(\mathcal{A})$ in the form $\alpha = \hat{P} - \hat{Q}$ for some objects $P, Q \in \mathcal{P}$, one defines $K_\alpha = [K_P] * [K_Q]^{-1}$, $K_\alpha^* = [K_P^*] * [K_Q^*]^{-1}$. Note that the equalities in (2.5) continue to hold with the elements $[K_P]$ and $[K_P^*]$ replaced by K_α and K_α^* , respectively, for any $\alpha \in K_0(\mathcal{A})$.

For each object $M \in \mathcal{A}$, by [1, Lemma 4.1], we fix a minimal projective resolution¹ of the form

$$0 \longrightarrow \Omega_M \xrightarrow{\delta_M} P_M \longrightarrow M \longrightarrow 0. \quad (2.8)$$

Set

$$C_M := \Omega_M \begin{array}{c} \xrightarrow{\delta_M} \\ \xleftarrow{0} \end{array} P_M.$$

Since the minimal projective resolution of M is unique up to isomorphism, the complex C_M is well-defined up to isomorphism.

Lemma 2.8. ([1, Lemma 4.2]) *Each object $L_\bullet \in \mathcal{C}_2(\mathcal{P})$ has a direct sum decomposition*

$$L_\bullet = C_M \oplus C_N^* \oplus K_P \oplus K_Q^*.$$

Moreover, the objects $M, N \in \mathcal{A}$ and $P, Q \in \mathcal{P}$ are uniquely determined up to isomorphism.

As in [1], we have an element E_M in $\mathcal{DH}(\mathcal{A})$ defined by

$$E_M := v^{\langle \hat{\Omega}_M, \hat{M} \rangle} \cdot K_{-\hat{\Omega}_M} * [C_M] \in \mathcal{DH}(\mathcal{A}).$$

It is easy to see that the shift functor defines an algebra involution $*$ on $\mathcal{DH}(\mathcal{A})$. Set $F_M = E_M^*$ for any object $M \in \mathcal{A}$.

Theorem 2.9. ([1, Lemmas 4.6, 4.7]) *The maps*

$$\begin{aligned} I_+^e : \tilde{\mathcal{H}}(\mathcal{A}) &\hookrightarrow \mathcal{DH}(\mathcal{A}), \quad [M] \mapsto E_M, \quad K_\alpha \mapsto K_\alpha; \\ I_-^e : \tilde{\mathcal{H}}(\mathcal{A}) &\hookrightarrow \mathcal{DH}(\mathcal{A}), \quad [M] \mapsto F_M, \quad K_\alpha \mapsto K_\alpha^* \end{aligned}$$

*are both embeddings of algebras. Moreover, the multiplication map $m : a \otimes b \mapsto I_+^e(a) * I_-^e(b)$ defines an isomorphism of vector spaces*

$$m : \tilde{\mathcal{H}}(\mathcal{A}) \otimes \tilde{\mathcal{H}}(\mathcal{A}) \xrightarrow{\simeq} \mathcal{DH}(\mathcal{A}).$$

¹The notations P_M and Ω_M will be used throughout the paper.

3. MAIN THEOREM

In this section, we first present the main theorem which was stated by Bridgeland in [1], proved by Yanagida in [17], and generalized by Lu and Peng in [5]. Then we provide a new and succinct proof by using the associativity formula of Hall algebras.

Main Theorem ([1],[17],[5]) *The Bridgeland's Hall algebra $\mathcal{DH}(\mathcal{A})$ is isomorphic to the Drinfeld double Hall algebra $D(\mathcal{A})$.*

In what follows, we will give the proof of Main Theorem. By Theorem 2.9, it suffices to prove that the commutator relation

$$\sum \varphi(a_{(2)}, b_{(1)}) I_+^e(a_{(1)}) * I_-^e(b_{(2)}) = \sum \varphi(a_{(1)}, b_{(2)}) I_-^e(b_{(1)}) * I_+^e(a_{(2)}) \quad (3.1)$$

holds in $\mathcal{DH}(\mathcal{A})$ for each $a = [A]K_\alpha$ and $b = [B]K_\beta$ with $\alpha, \beta \in K_0(\mathcal{A})$ and $[A], [B] \in \text{Iso}(\mathcal{A})$. By writing out the comultiplications $\Delta([A]K_\alpha)$ and $\Delta([B]K_\beta)$, and substituting into (3.1), we find that we only need to prove that Relation (3.1) holds for $a = [A]$ and $b = [B]$.

Since

$$\begin{aligned} \Delta([A]) &= \sum_{[A_1], [A_2]} v^{\langle A_1, A_2 \rangle} g_{A_1 A_2}^A [A_1] K_{\hat{A}_2} \otimes [A_2]; \\ \Delta([B]) &= \sum_{[B_1], [B_2]} v^{\langle B_1, B_2 \rangle} g_{B_1 B_2}^B [B_1] K_{\hat{B}_2} \otimes [B_2], \end{aligned}$$

the left hand side of (3.1) becomes

$$\begin{aligned} \text{LHS of (3.1)} &= \sum_{[A_1], [A_2], [B_1], [B_2]} v^{\langle A_1, A_2 \rangle + \langle B_1, B_2 \rangle} g_{A_1 A_2}^A g_{B_1 B_2}^B \varphi([A_2], [B_1] K_{\hat{B}_2}) E_{A_1} K_{\hat{A}_2} F_{B_2} \\ &= \sum_{[A_1], [A_2], [B_2]} v^{\langle A_1, A_2 \rangle + \langle A_2, B_2 \rangle} g_{A_1 A_2}^A g_{A_2 B_2}^B a_{A_2} E_{A_1} K_{\hat{A}_2} F_{B_2} \\ &= \sum_{[A_1], [A_2], [B_2]} v^{\langle A_1, A_2 \rangle + \langle A_2, B_2 \rangle} g_{A_1 A_2}^A g_{A_2 B_2}^B a_{A_2} v^{\langle \Omega_{A_1}, A_1 \rangle} K_{-\hat{\Omega}_{A_1}} [C_{A_1}] K_{\hat{A}_2} v^{\langle \Omega_{B_2}, B_2 \rangle} K_{-\hat{\Omega}_{B_2}}^* [C_{B_2}^*] \\ &= \sum_{[A_1], [A_2], [B_2]} v^x g_{A_1 A_2}^A g_{A_2 B_2}^B a_{A_2} K_{\hat{A}_2 - \hat{\Omega}_{A_1}} K_{-\hat{\Omega}_{B_2}}^* [C_{A_1}] [C_{B_2}^*], \end{aligned}$$

where

$$\begin{aligned} x &= \langle A_1, A_2 \rangle + \langle A_2, B_2 \rangle + \langle \Omega_{A_1}, A_1 \rangle + \langle \Omega_{B_2}, B_2 \rangle - (A_2, A_1) - (\Omega_{B_2}, A_1) \\ &= \langle \hat{A}_2 + \hat{\Omega}_{B_2}, \hat{B}_2 - \hat{A}_1 \rangle + \langle \hat{\Omega}_{A_1}, \hat{A}_1 \rangle - \langle \hat{A}_1, \hat{\Omega}_{B_2} \rangle. \end{aligned}$$

$$\begin{aligned} \text{RHS of (3.1)} &= \sum_{[A_1], [A_2], [B_1], [B_2]} v^{\langle A_1, A_2 \rangle + \langle B_1, B_2 \rangle} g_{A_1 A_2}^A g_{B_1 B_2}^B \varphi([A_1] K_{\hat{A}_2}, [B_2]) F_{B_1} K_{\hat{B}_2}^* E_{A_2} \\ &= \sum_{[A'_1], [A'_2], [B'_1]} v^{\langle A'_1, A'_2 \rangle + \langle B'_1, A'_1 \rangle} g_{A'_1 A'_2}^A g_{B'_1 A'_1}^B a_{A'_1} F_{B'_1} K_{\hat{A}'_1}^* E_{A'_2}. \end{aligned}$$

$$\begin{aligned}
&= \sum_{[A'_1], [A'_2], [B'_1]} v^{\langle A'_1, A'_2 \rangle + \langle B'_1, A'_1 \rangle} g_{A'_1 A'_2}^A g_{B'_1 A'_1}^B a_{A'_1} v^{\langle \Omega_{B'_1}, B'_1 \rangle} K_{-\hat{\Omega}_{B'_1}}^* [C_{B'_1}^*] K_{\hat{A}'_1}^* v^{\langle \Omega_{A'_2}, A'_2 \rangle} K_{-\hat{\Omega}_{A'_2}} [C_{A'_2}] \\
&= \sum_{[A'_1], [A'_2], [B'_1]} v^{x'} g_{A'_1 A'_2}^A g_{B'_1 A'_1}^B a_{A'_1} K_{\hat{A}'_1 - \hat{\Omega}_{B'_1}}^* K_{-\hat{\Omega}_{A'_2}} [C_{B'_1}^*] [C_{A'_2}],
\end{aligned}$$

where

$$\begin{aligned}
x' &= \langle A'_1, A'_2 \rangle + \langle B'_1, A'_1 \rangle + \langle \Omega_{B'_1}, B'_1 \rangle + \langle \Omega_{A'_2}, A'_2 \rangle - \langle A'_1, B'_1 \rangle - \langle \Omega_{A'_2}, B'_1 \rangle \\
&= \langle \hat{A}'_1 + \hat{\Omega}_{A'_2}, \hat{A}'_2 - \hat{B}'_1 \rangle + \langle \hat{\Omega}_{B'_1}, \hat{B}'_1 \rangle - \langle \hat{B}'_1, \hat{\Omega}_{A'_2} \rangle.
\end{aligned}$$

Lemma 3.1. *For any objects $X, Y \in \mathcal{A}$ and $T, W \in \mathcal{P}$. In $\mathcal{DH}(\mathcal{A})$ we have*

$$[C_X \oplus C_Y^* \oplus K_T \oplus K_W^*] = v^{\langle \hat{W} - \hat{T}, \hat{X} - \hat{Y} \rangle} K_{\hat{T}} K_{\hat{W}}^* [C_X \oplus C_Y^*].$$

Proof. By the commutative diagram

$$\begin{array}{ccc}
T \oplus W & \xrightleftharpoons[\begin{pmatrix} 0 & 1 \end{pmatrix}]{\begin{pmatrix} 1 & 0 \end{pmatrix}} & T \oplus W \\
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \downarrow & & \downarrow \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \\
\Omega_X \oplus P_Y & \xrightleftharpoons[\begin{pmatrix} 0 & -\delta_Y \end{pmatrix}]{\begin{pmatrix} \delta_X & 0 \end{pmatrix}} & P_X \oplus \Omega_Y,
\end{array}$$

we easily obtain that

$$|\mathrm{Hom}_{\mathcal{C}_2(\mathcal{A})}(K_T \oplus K_W^*, C_X \oplus C_Y^*)| = q^{\langle \hat{T}, \hat{\Omega}_X + \hat{P}_Y \rangle + \langle \hat{W}, \hat{P}_X + \hat{\Omega}_Y \rangle}.$$

Hence,

$$[K_T \oplus K_W^*][C_X \oplus C_Y^*] = v^m [C_X \oplus C_Y^* \oplus K_T \oplus K_W^*],$$

where

$$\begin{aligned}
m &= \langle \hat{T} + \hat{W}, \hat{P}_X + \hat{\Omega}_X + \hat{P}_Y + \hat{\Omega}_Y \rangle - 2\langle \hat{T}, \hat{\Omega}_X + \hat{P}_Y \rangle - 2\langle \hat{W}, \hat{P}_X + \hat{\Omega}_Y \rangle \\
&= \langle \hat{T} - \hat{W}, \hat{X} - \hat{Y} \rangle.
\end{aligned}$$

□

For any fixed objects $A, B, X, Y \in \mathcal{A}$, we denote by W_{AB}^{XY} the set

$$\{(f, g, h) \mid 0 \longrightarrow X \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} Y \longrightarrow 0 \text{ is exact in } \mathcal{A}\},$$

and denote by ${}_X\mathrm{Hom}_{\mathcal{A}}(A_1, B_2)_Y$ the set

$$\{g \mid 0 \longrightarrow X \longrightarrow A \xrightarrow{g} B \longrightarrow Y \longrightarrow 0 \text{ is exact in } \mathcal{A}\}.$$

By [13, (8.8)],

$$|W_{AB}^{XY}| = a_X a_Y \sum_{[L]} a_L g_{LX}^A g_{YL}^B,$$

and it is easy to see that

$$|{}_X\mathrm{Hom}_{\mathcal{A}}(A, B)_Y| = \frac{|W_{AB}^{XY}|}{a_X a_Y}. \quad (3.2)$$

Lemma 3.2. (1) *For any $A_1, B_2 \in \mathcal{A}$. In $\mathcal{DH}(\mathcal{A})$ we have*

$$[C_{A_1}][C_{B_2}^*] = \sum_{[X],[Y],[L]} v^a a_L g_{LX}^{A_1} g_{YL}^{B_2} K_{\hat{\Omega}_{A_1} - \hat{\Omega}_X} K_{\hat{P}_{B_2} - \hat{P}_Y}^* [C_X \oplus C_Y^*],$$

where

$$a = \langle P_{A_1}, \Omega_{B_2} \rangle - \langle \Omega_{A_1}, P_{B_2} \rangle + \langle \hat{P}_{B_2} + \hat{\Omega}_X - \hat{\Omega}_{A_1} - \hat{P}_Y, \hat{X} - \hat{Y} \rangle;$$

(2) *For any $A_2, B_1 \in \mathcal{A}$. In $\mathcal{DH}(\mathcal{A})$ we have*

$$[C_{B_1}^*][C_{A_2}] = \sum_{[X],[Y],[L']} v^{a'} a_{L'} g_{L'Y}^{B_1} g_{XL'}^{A_2} K_{\hat{P}_{A_2} - \hat{P}_X} K_{\hat{\Omega}_{B_1} - \hat{\Omega}_Y}^* [C_X \oplus C_Y^*],$$

where

$$a' = \langle P_{B_1}, \Omega_{A_2} \rangle - \langle \Omega_{B_1}, P_{A_2} \rangle + \langle \hat{P}_X + \hat{\Omega}_{B_1} - \hat{P}_{A_2} - \hat{\Omega}_Y, \hat{X} - \hat{Y} \rangle.$$

Proof. We only prove (1), since (2) is similar.

$$\begin{aligned} \mathrm{Ext}_{\mathcal{C}_2(\mathcal{A})}^1(C_{A_1}, C_{B_2}^*) &\cong \mathrm{Hom}_{\mathcal{K}_2(\mathcal{P})}(C_{A_1}, C_{B_2}) \\ &\cong \mathrm{Hom}_{\mathcal{R}_2(\mathcal{A})}(A_1, B_2) \\ &\cong \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{D^b(\mathcal{A})}(A_1, B_2[2i]) \\ &\cong \mathrm{Hom}_{\mathcal{A}}(A_1, B_2), \text{ since } \mathcal{A} \text{ is hereditary.} \end{aligned} \quad (3.3)$$

Consider an extension of C_{A_1} by $C_{B_2}^*$

$$\eta : 0 \longrightarrow C_{B_2}^* \longrightarrow L_\bullet \longrightarrow C_{A_1} \longrightarrow 0.$$

It induces a long exact sequence in homology

$$H_0(C_{B_2}^*) \longrightarrow H_0(L_\bullet) \longrightarrow H_0(C_{A_1}) \longrightarrow H_1(C_{B_2}^*) \longrightarrow H_1(L_\bullet) \longrightarrow H_1(C_{A_1}).$$

Writing $L_\bullet = C_X \oplus C_Y^* \oplus K_T \oplus K_W^*$ for some objects $X, Y \in \mathcal{A}$ and $T, W \in \mathcal{P}$, we obtain the following exact sequence

$$0 \longrightarrow X \longrightarrow A_1 \xrightarrow{\delta} B_2 \longrightarrow Y \longrightarrow 0,$$

where δ is determined by the equivalence class of η via the canonical isomorphism in (3.3)

$$\mathrm{Ext}_{\mathcal{C}_2(\mathcal{A})}^1(C_{A_1}, C_{B_2}^*) \cong \mathrm{Hom}_{\mathcal{A}}(A_1, B_2). \quad (3.4)$$

By considering the kernels of differentials in $C_{B_2}^*, L_\bullet$ and C_{A_1} , we get that

$$P_Y \oplus W \cong P_{B_2}, \text{ and then } \Omega_X \oplus T \cong \Omega_{A_1}.$$

That is, T and W are uniquely determined by X and Y up to isomorphism, respectively. The canonical isomorphism (3.4) induces an isomorphism

$$\mathrm{Ext}_{\mathcal{C}_2(\mathcal{A})}^1(C_{A_1}, C_{B_2}^*)_{C_X \oplus C_Y^* \oplus K_T \oplus K_W^*} \cong {}_X\mathrm{Hom}_{\mathcal{A}}(A_1, B_2)_Y.$$

Hence,

$$|\mathrm{Ext}_{\mathcal{C}_2(\mathcal{A})}^1(C_{A_1}, C_{B_2}^*)_{C_X \oplus C_Y^* \oplus K_T \oplus K_W^*}| = \frac{|W_{A_1 B_2}^{XY}|}{a_X a_Y} = \sum_{[L]} a_L g_{LX}^{A_1} g_{YL}^{B_2}.$$

Thus,

$$[C_{A_1}][C_{B_2}^*] = \sum_{[X],[Y],[L]} v^{\langle P_{A_1}, \Omega_{B_2} \rangle - \langle \Omega_{A_1}, P_{B_2} \rangle} a_L g_{LX}^{A_1} g_{YL}^{B_2} [C_X \oplus C_Y^* \oplus K_T \oplus K_W^*],$$

here we have used that

$$|\mathrm{Hom}_{\mathcal{C}_2(\mathcal{A})}(C_{A_1}, C_{B_2}^*)| = q^{\langle \Omega_{A_1}, P_{B_2} \rangle}.$$

By Lemma 3.1,

$$\begin{aligned} [C_{A_1}][C_{B_2}^*] &= \sum_{[X],[Y],[L]} v^a a_L g_{LX}^{A_1} g_{YL}^{B_2} K_T K_W^* [C_X \oplus C_Y^*] \\ &= \sum_{[X],[Y],[L]} v^a a_L g_{LX}^{A_1} g_{YL}^{B_2} K_{\hat{\Omega}_{A_1} - \hat{\Omega}_X} K_{\hat{P}_{B_2} - \hat{P}_Y}^* [C_X \oplus C_Y^*], \end{aligned}$$

where

$$\begin{aligned} a &= \langle P_{A_1}, \Omega_{B_2} \rangle - \langle \Omega_{A_1}, P_{B_2} \rangle + \langle \hat{W} - \hat{T}, \hat{X} - \hat{Y} \rangle \\ &= \langle P_{A_1}, \Omega_{B_2} \rangle - \langle \Omega_{A_1}, P_{B_2} \rangle + \langle \hat{P}_{B_2} + \hat{\Omega}_X - \hat{\Omega}_{A_1} - \hat{P}_Y, \hat{X} - \hat{Y} \rangle. \end{aligned}$$

□

Proof of Main Theorem

LHS of (3.1)

$$\begin{aligned} &= \sum_{\substack{[A_1],[A_2],[B_2], \\ [L],[X],[Y]}} v^{a+x} g_{A_1 A_2}^A g_{A_2 B_2}^B g_{LX}^{A_1} g_{YL}^{B_2} a_{A_2} a_L K_{\hat{A}_2 - \hat{\Omega}_{A_1}} K_{-\hat{\Omega}_{B_2}}^* K_{\hat{\Omega}_{A_1} - \hat{\Omega}_X} K_{\hat{P}_{B_2} - \hat{P}_Y}^* [C_X \oplus C_Y^*] \\ &= \sum_{\substack{[A_1],[A_2],[B_2], \\ [L],[X],[Y]}} v^{\langle \hat{B}_2 + \hat{\Omega}_X - \hat{P}_Y - \hat{A}_2, \hat{X} - \hat{Y} \rangle} g_{A_1 A_2}^A g_{A_2 B_2}^B g_{LX}^{A_1} g_{YL}^{B_2} a_{A_2} a_L K_{\hat{A}_2 - \hat{\Omega}_X} K_{\hat{B}_2 - \hat{P}_Y}^* [C_X \oplus C_Y^*] \\ &= \sum_{\substack{[A_1],[A_2],[B_2], \\ [L],[X],[Y]}} v^{\langle \hat{Y} + \hat{L} + \hat{\Omega}_X - \hat{P}_Y - \hat{A}_2, \hat{X} - \hat{Y} \rangle} g_{A_1 A_2}^A g_{A_2 B_2}^B g_{LX}^{A_1} g_{YL}^{B_2} a_{A_2} a_L K_{\hat{A}_2 - \hat{\Omega}_X} K_{\hat{Y} + \hat{L} - \hat{P}_Y}^* [C_X \oplus C_Y^*] \\ &= \sum_{\substack{[A_2],[L], \\ [X],[Y]}} v^{\langle \hat{L} + \hat{\Omega}_X - \hat{\Omega}_Y - \hat{A}_2, \hat{X} - \hat{Y} \rangle} g_{LX A_2}^A g_{A_2 Y L}^B a_{A_2} a_L K_{\hat{A}_2 - \hat{\Omega}_X} K_{\hat{L} - \hat{\Omega}_Y}^* [C_X \oplus C_Y^*], \end{aligned}$$

here we get the last equality by using the associativity formula in (2.1).

LHS of (3.1)

$$\begin{aligned}
&= \sum_{\substack{[A'_1], [A'_2], [B'_1], \\ [L'], [X], [Y]}} v^{a'+x'} g_{A'_1 A'_2}^A g_{B'_1 A'_1}^B g_{L' Y}^{B'_1} g_{X L'}^{A'_2} a_{A'_1} a_{L'} K_{\hat{A}'_1 - \hat{\Omega}_{B'_1}}^* K_{-\hat{\Omega}_{A'_2}} K_{\hat{P}_{A'_2} - \hat{P}_X} K_{\hat{\Omega}_{B'_1} - \hat{\Omega}_Y}^* [C_X \oplus C_Y^*] \\
&= \sum_{\substack{[A'_1], [A'_2], [B'_1], \\ [L'], [X], [Y]}} v^{\langle \hat{A}'_1 + \hat{P}_X - \hat{\Omega}_Y - \hat{A}'_2, \hat{X} - \hat{Y} \rangle} g_{A'_1 A'_2}^A g_{B'_1 A'_1}^B g_{L' Y}^{B'_1} g_{X L'}^{A'_2} a_{A'_1} a_{L'} K_{\hat{A}'_2 - \hat{P}_X} K_{\hat{A}'_1 - \hat{\Omega}_Y}^* [C_X \oplus C_Y^*] \\
&= \sum_{\substack{[A'_1], [A'_2], [B'_1], \\ [L'], [X], [Y]}} v^{\langle \hat{A}'_1 + \hat{P}_X - \hat{\Omega}_Y - \hat{X} - \hat{L}', \hat{X} - \hat{Y} \rangle} g_{A'_1 A'_2}^A g_{B'_1 A'_1}^B g_{L' Y}^{B'_1} g_{X L'}^{A'_2} a_{A'_1} a_{L'} K_{\hat{X} + \hat{L}' - \hat{P}_X} K_{\hat{A}'_1 - \hat{\Omega}_Y}^* [C_X \oplus C_Y^*] \\
&= \sum_{\substack{[L'], [A'_1], \\ [X], [Y]}} v^{\langle \hat{A}'_1 + \hat{\Omega}_X - \hat{\Omega}_Y - \hat{L}', \hat{X} - \hat{Y} \rangle} g_{A'_1 X L'}^A g_{L' Y A'_1}^B a_{L'} a_{A'_1} K_{\hat{L}' - \hat{\Omega}_X} K_{\hat{A}'_1 - \hat{\Omega}_Y}^* [C_X \oplus C_Y^*].
\end{aligned}$$

Identifying $[L]$ and $[A_2]$ in LHS of (3.1) with $[A'_1]$ and $[L']$ in RHS of (3.1), respectively, we obtain that

$$\text{LHS of (3.1)} = \text{RHS of (3.1)}.$$

□

4. APPENDIX: A REMARK ON MODIFIED RINGEL–HALL ALGEBRAS

In this section, we briefly give a proof of Main Theorem 4.11 in [5] by using the associativity formula of Hall algebras without [5, Lemma 4.10]. Let \mathcal{A} be a finitary hereditary abelian k -category, and we do not assume that it has enough projectives. Inspired by the works of Bridgeland [1] and Gorsky [2], Lu and Peng [5] introduced an algebra $\mathcal{MH}_{\mathbb{Z}/2, tw}(\mathcal{A})$, called the *modified Ringel–Hall algebra* of \mathcal{A} , with the purpose of generalizing Bridgeland's construction to any hereditary abelian categories satisfying certain finiteness conditions. For unexplained notations (such as C_X, C_Y^*, K_X , and K_Y^*) concerning the modified Ringel–Hall algebra we refer to [5].

For any $A, B \in \mathcal{A}$, we consider the action of the group $\text{Aut}_{\mathcal{A}}(A) \times \text{Aut}_{\mathcal{A}}(B)$ on $\text{Hom}_{\mathcal{A}}(A, B)$, which is defined by the following commutative diagram

$$\begin{array}{ccc}
A & \xrightarrow{g} & B \\
s \downarrow & & \downarrow t \\
A & \xrightarrow{g'} & B,
\end{array}$$

for any $g \in \text{Hom}_{\mathcal{A}}(A, B)$, $s \in \text{Aut}_{\mathcal{A}}(A)$ and $t \in \text{Aut}_{\mathcal{A}}(B)$. We denote by O_g the orbit of g for any $g \in \text{Hom}_{\mathcal{A}}(A, B)$, and set $O = \{O_g \mid g \in \text{Hom}_{\mathcal{A}}(A, B)\}$. Clearly, we have the

following equations

$$\sum_{\substack{O_g \in O: g \in \text{Hom}_{\mathcal{A}}(A, B) \\ \ker g \cong X \\ \text{coker } g \cong Y}} |O_g| = |{}_X \text{Hom}_{\mathcal{A}}(A, B)_Y| = \sum_{[L]} a_L g_{LX}^A g_{YL}^B. \quad (4.1)$$

By (4.1), we reformulate [5, Lemma 4.9] as follows.

Lemma 4.1. ([5]) (1) *For any $A_1, B_2 \in \mathcal{A}$. In $\mathcal{MH}_{\mathbb{Z}/2, tw}(\mathcal{A})$ we have*

$$[C_{A_1}] * [C_{B_2}^*] = \sum_{[X], [Y], [L]} v^{\langle \hat{Y} - \hat{X}, \hat{B}_2 - \hat{Y} \rangle} a_L g_{LX}^{A_1} g_{YL}^{B_2} [C_X \oplus C_Y^*] * [K_{\hat{B}_2 - \hat{Y}}^*];$$

(2) *For any $A_2, B_1 \in \mathcal{A}$. In $\mathcal{MH}_{\mathbb{Z}/2, tw}(\mathcal{A})$ we have*

$$[C_{B_1}^*] * [C_{A_2}] = \sum_{[X], [Y], [L']} v^{\langle \hat{X} - \hat{Y}, \hat{A}_2 - \hat{X} \rangle} a_{L'} g_{L'Y}^{B_1} g_{XL'}^{A_2} [C_X \oplus C_Y^*] * [K_{\hat{A}_2 - \hat{X}}].$$

Remark 4.2. In Lemma 4.1, we have employed the additive Euler form rather than the multiplicative one used in [5].

Using Lemma 4.1, we simplify the proof of [5, Thm. 4.11] as follows.

LHS of the equation (19) in [5]

$$\begin{aligned} &= \sum_{[A_1], [A_2], [B_2]} v^{\langle A_1, A_2 \rangle + \langle A_2, B_2 \rangle - \langle A_2, B_2 \rangle} g_{A_1 A_2}^A g_{A_2 B_2}^B a_{A_2} [C_{A_1}] * [C_{B_2}^*] * K_{\hat{A}_2} \\ &= \sum_{\substack{[A_1], [A_2], [B_2], \\ [X], [Y], [L]}} v^{\langle \hat{A}_1 - \hat{B}_2, \hat{A}_2 \rangle + \langle \hat{Y} - \hat{X}, \hat{B}_2 - \hat{Y} \rangle} g_{A_1 A_2}^A g_{A_2 B_2}^B g_{LX}^{A_1} g_{YL}^{B_2} a_{A_2} a_L [C_X \oplus C_Y^*] * K_{\hat{B}_2 - \hat{Y}}^* * K_{\hat{A}_2} \\ &\quad \frac{\hat{A}_1 = \hat{L} + \hat{X}}{\hat{B}_2 = \hat{Y} + \hat{L}} \sum_{\substack{[A_2], [L], \\ [X], [Y]}} v^{\langle \hat{X} - \hat{Y}, \hat{A}_2 - \hat{L} \rangle} g_{LX A_2}^A g_{A_2 YL}^B a_{A_2} a_L [C_X \oplus C_Y^*] * K_{\hat{L}}^* * K_{\hat{A}_2}, \end{aligned}$$

here we have used the associativity formula in (2.1).

RHS of the equation (19) in [5]

$$\begin{aligned} &= \sum_{[\tilde{A}_1], [\tilde{A}_2], [\tilde{B}_1]} v^{\langle \tilde{A}_1, \tilde{A}_2 \rangle + \langle \tilde{B}_1, \tilde{A}_1 \rangle - \langle \tilde{A}_1, \tilde{A}_2 \rangle} g_{\tilde{A}_1 \tilde{A}_2}^A g_{\tilde{B}_1 \tilde{A}_1}^B a_{\tilde{A}_1} [C_{\tilde{B}_1}^*] * [C_{\tilde{A}_2}] * K_{\hat{\tilde{A}}_1}^* \\ &= \sum_{\substack{[\tilde{A}_1], [\tilde{A}_2], [\tilde{B}_1], \\ [X], [Y], [L']}} v^{\langle \hat{\tilde{B}}_1 - \hat{\tilde{A}}_2, \hat{\tilde{A}}_1 \rangle + \langle \hat{X} - \hat{Y}, \hat{\tilde{A}}_2 - \hat{X} \rangle} g_{\tilde{A}_1 \tilde{A}_2}^A g_{\tilde{B}_1 \tilde{A}_1}^B g_{L'Y}^{\tilde{B}_1} g_{XL'}^{\tilde{A}_2} a_{L'} a_{\tilde{A}_1} [C_X \oplus C_Y^*] * K_{\hat{\tilde{A}}_2 - \hat{X}}^* * K_{\hat{\tilde{A}}_1}^* \\ &\quad \frac{\hat{\tilde{B}}_1 = \hat{L}' + \hat{Y}}{\hat{\tilde{A}}_2 = \hat{X} + \hat{L}'} \sum_{\substack{[\tilde{A}_1], [L'], \\ [X], [Y]}} v^{\langle \hat{X} - \hat{Y}, \hat{L}' - \hat{\tilde{A}}_1 \rangle} g_{\tilde{A}_1 X L'}^A g_{L'Y \tilde{A}_1}^B a_{L'} a_{\tilde{A}_1} [C_X \oplus C_Y^*] * K_{\hat{\tilde{A}}_1}^* * K_{\hat{L}'}. \end{aligned}$$

Identifying $[L]$ and $[A_2]$ in LHS with $[\tilde{A}_1]$ and $[L']$ in RHS, respectively, we obtain that

$$\text{LHS of the equation (19) in [5]} = \text{RHS of the equation (19) in [5]}.$$

Remark 4.3. The preliminary part (Lemma 3.2) for the proof of Main Theorem is similar to [5, Lemma 4.9], but the calculation methods are not the same. In this note, we explicitly work out the coefficients in the summation via Hall numbers. While, Lu and Peng introduced orbit sets O_g and express the coefficients by $|O_g|$. For the conclusive part, we are reduced to use the associativity formula of Hall algebras, and Lu and Peng introduced once more two sets and give a characterization of the cardinalities of these sets ([5, Lemma 4.10]), then they are reduced to the equality of the cardinalities of these two sets. In some sense, we avoid computing the cardinalities of the sets defined by Lu and Peng, this work seems to be equivalent to the proof of the associativity of Hall algebras. However, the advantage of their proof is that we have better understanding of the essence of the coefficients in the commutator relation (3.1).

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